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TENSOR PRODUCTS OF PRINCIPAL SERIES FOR THE DESITTER GROUP[1]

BY

ROBERT P. MARTIN

ABSTRACT. The decomposition of the tensor product of two principal series representations is determined for the simply connected double covering, $G = \text{Spin}(4, 1)$, of the DeSitter group. The main result is that this decomposition consists of two pieces, T_c and T_d, where T_c is a continuous direct sum with respect to Plancherel measure on $\hat{G}$ of representations from the principal series only and T_d is a discrete sum of representations from the discrete series of G. The multiplicities of representations occurring in T_c and T_d are all finite.

Introduction. Let $G = \text{Spin}(4, 1)$ be the simply connected double covering of the DeSitter group, $G = KAN$ an Iwasawa decomposition of G, M the centralizer of A in K, and $P = MAN$ the associated minimal parabolic subgroup of G. For $\sigma \in \hat{M}$ and $\tau \in \hat{A}$, $\sigma \times \tau$ is a representation of P via $\sigma \times \tau(man) = \sigma(m)\tau(a)$ and a representation of the form $\pi(\sigma, \tau) = \text{Ind}_P^G\, \sigma \times \tau$ is called a principal series representation of G. The main goal of this paper is to determine the decomposition of the tensor product of two principal series representations of G into irreducibles.

It was shown in [7], by using Mackey's tensor product theorem and the Mackey-Anh reciprocity theorem, that this problem reduces to knowing how to decompose the restriction to MA of almost every principal series representation of G and each discrete series representation of G. For a representation π belonging to the principal series of G, the restriction of π to MA, $(\pi)_{MA}$, was determined by using Mackey's subgroup theorem. However, in that paper, we were not able to determine explicitly $(\pi)_{MA}$ for a representation π belonging to the discrete series of G. This we do in §3 of this paper by using Lie algebraic methods and the realizations of these representations given by Dixmier in [2].

This paper is organized as follows. In §§1 and 2 we summarize the main results concerning the structure and representation theory of G that we shall use. In §3 we determine $(\pi)_{MA}$ when π is a discrete series representation of G. We also include the results of [7] concerning the decomposition of $(\pi)_{MA}$ when π is a principal series representation of G. In §4 we show how to decompose the tensor product of two principal series representations of G. The main results are contained in Theorem 4.

The basic methodology used in this paper to decompose principal series tensor products originates in the works of G. Mackey [6], N. Anh [1], and F. Williams [11].

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1. The structure of G. In this section we summarize the main results concerning the structure of the DeSitter group and its two-fold covering Spin(4, 1) that we shall use in this paper. Further details may be found in [**8**].

Let $O(4, 1)$ denote the group of linear transformations of $\mathbf{R}^5$ which preserve the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$. If J is the diagonal matrix $J = [-1, 1, \ldots, 1]$, then $O(4, 1)$ may be identified with $\{g \in GL(5, \mathbf{R}) \colon {}^t gJg = J\}$. The connected component of the identity is the group

$$G' = SO_e(4, 1) = \{g \in O(4, 1)\colon \det(g) = 1, g_{00} > 1\}$$

which is commonly referred to as the DeSitter group. G' is a connected semisimple real-rank one Lie group with trivial center. We let $G = \text{Spin}(4, 1)$ denote the simply-connected double covering of G'. As indicated in [**8**], we may realize G as a certain collection of two-by-two matrices over the quaternions. If

$$F = \{x_1 + ix_2 + jx_3 + kx_4\colon x_i \in \mathbf{R}, i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k, jk = -kj = i, ki = -ik = j\}$$

denotes the quaternions, $\bar{x} = x_1 - ix_2 - jx_3 - kx_4$, and $|x| = \sqrt{x\bar{x}}$, then G is isomorphic to the group

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\colon a, b, c, d \in F, \bar{a}b = \bar{c}d, |a|^2 - |c|^2 = 1, |d|^2 - |b|^2 = 1\right\}.$$

The group $U = \{x \in F\colon |x| = 1\}$ of unit quaternions is easily seen to be isomorphic to $SU(2)$ [via the mapping

$$x_1 + ix_2 + jx_3 + kx_4 \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad a = x_1 + ix_2, \quad b = x_3 + ix_4,$$

for example], and if we let

$$K = \left\{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\colon u, v \in U\right\} \approx SU(2) \times SU(2) \approx \text{Spin}(4),$$

$$A = \left\{\begin{bmatrix} ch\tfrac{1}{2}t & sh\tfrac{1}{2}t \\ sh\tfrac{1}{2}t & ch\tfrac{1}{2}t \end{bmatrix} = a_t\colon t \in \mathbf{R}\right\} \approx \mathbf{R}^+,$$

and

$$N = \left\{\begin{pmatrix} 1 - x & x \\ -x & 1 + x \end{pmatrix} = n_x\colon x = \frac{1}{2}(x_2 i + x_3 j + x_4 k)\right\} \approx \mathbf{R}^3,$$

then $G = KAN$ is an Iwasawa decomposition of G. If M denotes the centralizer of A in K, then

$$M = \left\{\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} = m_u\colon u \in U\right\} \approx \text{Spin}(3)$$

and $P = MAN$ is a (minimal parabolic) subgroup of G which contains N as a normal subgroup. One easily computes that the actions of M and A on N are given by: $m_u n_x m_u^{-1} = m_u \cdot n_x = n_{ux\bar{u}}$, $a_t n_x a_t^{-1} = a_t \cdot n_x = n_{e^t x}$, i.e., M acts by rotations and A acts by dilations. If $\tilde{M}$ denotes the normalizer of A in K, then the Weyl group $W = \tilde{M}/M$ has order 2 and we may take $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ as representatives of the cosets of W. In addition to the Iwasawa decomposition of G, one has the KAK decomposition (see [**8**, p. 366]) and the Bruhat decomposition

$G = PeP \cup PwP$ (and so there are only two $P : P$ double cosets in G with only one of positive Haar measure). Setting

$$V = \left\{ \begin{pmatrix} 1 - x & -x \\ x & 1 + x \end{pmatrix} = v_x : x = \frac{1}{2}(ix_2 + jx_3 + kx_4) \right\}$$

and using the relations $w^{-1}Aw = A$, $w^{-1}Mw = M$, the latter decomposition can be expressed as $G = Pw^{-1} \cup PV$ and so up to a manifold of lower dimension (and so a set of Haar measure zero) we see that $G = PV$.

If $\phi: G \to G'$ denotes the homomorphism on p. 366 of [8] and we let $K' = \phi(K)$, $A' = \phi(A)$, $N' = \phi(N)$, $M' = \phi(M)$, and $V' = \phi(V)$, then

$$\ker(\phi) = \text{the center of } G = Z(G) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \subseteq K$$

and so we have the following isomorphisms:

$$K/Z(G) \approx K' \approx SO(4),\ M/Z(G) \approx M' \approx SO(3),$$
$$A \approx A' \approx \mathbf{R}^+,\ N \approx N' \approx V \approx V' \approx \mathbf{R}^3.$$

It is easily seen that

$$K' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} : k \in SO(4) \right\},$$

$$A' = \left\{ \begin{bmatrix} cha & sha & 0 \\ sha & cha & 0 \\ 0 & 0 & I \end{bmatrix} : a \in \mathbf{R} \right\},$$

$$M' = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u \end{bmatrix} : u \in SO(3) \right\}.$$

The Lie algebra $\mathfrak{g}$ of G (and of G') is the collection of matrices of the form

$$M(a, b, \ldots, k) = \begin{bmatrix} 0 & a & b & c & d \\ a & 0 & e & f & g \\ b & -e & 0 & h & j \\ c & -f & -h & 0 & k \\ d & -g & -j & -k & 0 \end{bmatrix} \quad \text{where } a, b, \ldots, k \in \mathbf{R}.$$

Letting $A = M(1, 0, \ldots, 0)$, $B = M(0, 1, 0, \ldots, 0), \ldots,$ $K = M(0, \ldots, 0, 1)$, we see that $A, B, \ldots, K$ forms a basis for $\mathfrak{g}$. If $\mathfrak{k} \subseteq \mathfrak{g}$ denotes the set of matrices of the form $eE + fF + gG + hH + jJ + kK$ and $\mathfrak{p} \subseteq \mathfrak{g}$ denotes the set of matrices of the form $aA + bB + cC + dD$, then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ and the associated Cartan involution θ is just the negative transpose. The subspace $\mathfrak{a} = \mathbf{R}A$ is a maximal abelian subalgebra of $\mathfrak{p}$. If $\alpha \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{R})$ is given by $\alpha(aA) = \alpha a$ and we define

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g}: [H, X] = \alpha(H)X, \text{for all } H \in \mathfrak{a} \},$$

then the nonzero weight spaces are

$$\mathfrak{g}_0 = \mathbf{R}A + \mathbf{R}H + \mathbf{R}J + \mathbf{R}K,$$
$$\mathfrak{g}_1 = \mathbf{R}(B + E) + \mathbf{R}(C + F) + \mathbf{R}(D + G) \quad \text{and}$$
$$\mathfrak{g}_{-1} = \mathbf{R}(-B + E) + \mathbf{R}(-C + F) + \mathbf{R}(-D + G).$$

We let $\mathfrak{n} = \mathfrak{g}_1$, $\mathfrak{v} = \mathfrak{g}_{-1}$ and $\mathfrak{m} = \mathbf{R}H + \mathbf{R}J + \mathbf{R}K$. It is easy to check that the Lie algebras of K, M, A, N, and V are, respectively, $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{v}$.

We denote by $\mathfrak{U}$ the universal enveloping algebra of $\mathfrak{g}$ and by $\mathfrak{g}^{\mathbf{C}}$ the complexification ($\approx so(5, \mathbf{C})$) of $\mathfrak{g}$. Basis elements and a multiplication table for $\mathfrak{g}^{\mathbf{C}}$ may be found in [**2**].

2. The representation theory of G. We shall use the classification of the irreducible representations of G given by Dixmier in [**2**]. Dixmier's techniques are very algebraic and rely on the one-to-one correspondence between equivalence classes of irreducible unitary representations of G and certain infinitesimal equivalence classes of algebraically $\mathfrak{k}$-finite representations of $\mathfrak{g}$ which are infinitesimally unitary (see [**2**] or [**10**, Vol. I, p. 330]). A more global treatment using induced representations may be found in [**8**].

We begin by looking at the representation theories of M, K, A, and $C = MA$. $M = \mathrm{Spin}(3) \approx SU(2)$ and it is well known that $\hat{M} = \{\sigma^k : k = 0, \frac{1}{2}, 1, \dots\}$ where each σ^k acts on a space V^k of dimension $2k + 1$ (see p. 110 of [**9**]). For $k = 0, 1, \dots,$ σ^k is a single-valued representation of $M' \approx SO(3)$ while each σ^k, $k = \frac{1}{2}, 3/2, \dots,$ is a double-valued representation of M'.

Since $K = \mathrm{Spin}(3) \times \mathrm{Spin}(3) \approx \mathrm{Spin}(4)$, we have that $\hat{K} = \{\sigma^{k,k'} \equiv \sigma^k \times \sigma^{k'}: k, k' = 0, \frac{1}{2}, 1, \dots\}$ where each $\sigma^{k,k'}$ acts on $V^{k,k'} \equiv V^k \times V^{k'}$. It is also known that the restriction of $\sigma^{k,k'}$ to M, $(\sigma^{k,k'})_M$, decomposes as

$$(\sigma^{k,k'})_M \simeq \bigoplus_{j=|k-k'|}^{k+k'} \sigma^j \qquad [\mathbf{9}, \text{p. } 175].$$

The representation $\sigma^{k,k'}$ is a single- or double-valued representation of K' according to whether $k + k'$ is integral or not.

The irreducible representations (quasi-characters) of A are given by $\lambda^s(a_t) = e^{st}$ for $s \in \mathbf{C}$. These are unitary precisely when $\mathrm{Re}(s) = 0$ and so we shall view $\hat{A} = \{\lambda^{iy}: y \in \mathbf{R}\}$. Since the group $C = MA$ is a direct product, $\hat{C} = \hat{M} \times \hat{A}$ with Plancherel measure μ_C on $\hat{C}$ being the product of the Plancherel measures on $\hat{M}$ and $\hat{A}$.

If $\rho \in \hat{G}$ acting on H_ρ, then Dixmier [**2**] shows that the restriction of ρ to K, $(\rho)_K$, contains a given $\sigma^{k,k'}$ with multiplicity 0 or 1. Letting Γ be the set of pairs (k, k') which occur in $(\rho)_K$, we then have that $(\rho)_K \simeq \bigoplus_\Gamma \sigma^{k,k'}$ and $H_\rho = \bigoplus_\Gamma V^{k,k'}$ (the Hilbert space direct sum). Furthermore, if

$$\Gamma_0 = \{(k, k'): k, k' = 0, \tfrac{1}{2}, 1, \dots\},$$
$$\Gamma_1 = \{(k, k') \in \Gamma_0: k + k' \equiv 0 \ (\mathrm{mod}\ 1)\},$$
$$\Gamma_2 = \{(k, k') \in \Gamma_0: k + k' \equiv \tfrac{1}{2} \ (\mathrm{mod}\ 1)\},$$

then $\Gamma_0 = \Gamma_1 \cup \Gamma_2$, $\Gamma \subseteq \Gamma_1$ or $\Gamma \subseteq \Gamma_2$, and Γ must consist of all points in Γ_1 or Γ_2 which lie on and within a region of the following form:

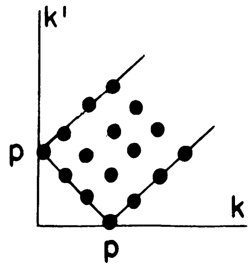

FIGURE 1

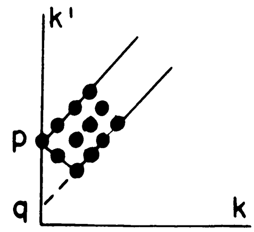

FIGURE 2

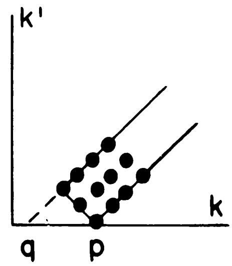

FIGURE 3

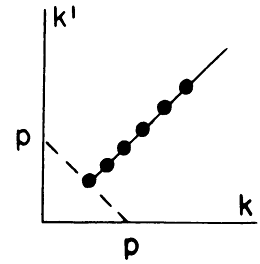

FIGURE 4

Using the same letters to denote the corresponding representations on the Lie algebras of G and K, we have that ρ, as a representation of $\mathfrak{g}$, acts on the space $H'_\rho = \Sigma_\Gamma V^{k,k'}$ (the algebraic sum) and that the restriction of ρ to $\mathfrak{k}$ acts on $V^{k,k'}$ as $\sigma^{k,k'}$. Dixmier then shows that by starting with a Γ of the above form and a basis for each $V^{k,k'}$, $(k, k') \in \Gamma$, it is possible to use a "closest neighbors" technique to extend the above action of ρ on $\mathfrak{k}$ in a compatible way to all of $\mathfrak{g}$ (in possibly many ways in the case of Figure 1) so as to obtain an irreducible representation of $\mathfrak{g}$. Thus in Dixmier's classification of $\hat{G}$, the irreducibles, other than the trivial representation, fall into four categories.

(A). *The representations* $\nu_{p,\sigma}$. These are the representations arising from Figure 1. To each half-integer p one gets a one parameter family of irreducibles indexed by σ. The range of σ is as follows: if $p = 0$, $\sigma > -2$; if $p = 1, 2, 3, \ldots$, $\sigma > 0$; while if $p = \frac{1}{2}, 3/2, \ldots$, $\sigma > \frac{1}{4}$. These representations arise quite naturally within the framework of induced representations. They constitute what are usually called the principal series (essentially for $\sigma \geqslant \frac{1}{4}$) and complementary series of irreducible unitary representations of G. If $\sigma^p \in \hat{M}$ and λ^s is a quasi-character of A, we let $(\sigma^p \times \lambda^s)'$ denote the representation of P given by $(\sigma^p \times \lambda^s)'(man) = \sigma^p(m)\lambda^s(a)$ and set $\pi(p, s) = \mathrm{Ind}_P^G(\sigma^p \times \lambda^s)'$. When $\mathrm{Re}(s) = 0$, the representation $\pi(p, s)$ is unitary and it is known that $\pi(p, s)$ is reducible iff $s = 0$ and $p = \frac{1}{2}, 3/2, \ldots$ (see [4]). For $p = \frac{1}{2}, 3/2, \ldots$, $\pi(p, 0)$ splits into the direct sum of 2 irreducibles which we shall denote by $\pi^{\pm}_{p,1/2}$ [8, pp. 390, 421]. It is well known that $\pi(p, iy) \simeq \pi(p, -iy)$ and that $\pi(p, iy) \not\simeq \pi(q, ix)$ if $p \neq q$ or $|y| \neq |x|$. The collection $\hat{G}_p = \{\pi(p, iy)$: $y \geqslant 0, p = 0, \frac{1}{2}, \ldots\}$ is called the principal series of G. We will write

$$\hat{G}_i = \left\{\pi(p, iy) \in \hat{G}_p : y \neq 0 \text{ if } p = \tfrac{1}{2}, \tfrac{3}{2}, \ldots\right\}$$

for the irreducible principal series and

$$\hat{G}_r = \left\{ \pi^{\pm}_{p,1/2} : p = \tfrac{1}{2}, \tfrac{3}{2}, \dots \right\}$$

for the collection of irreducibles arising from the reducible principal series representations. Finally, the representation $\pi(p, iy) \in \hat{G}_p$ corresponds to $\nu_{p,\sigma}$ in Dixmier's classification with $\sigma = \frac{1}{4} + y^2$.

When $\operatorname{Re}(s) \neq 0$, the representation $\pi(p, s)$ is not unitary. However, the results of [**4**] show that when $p = 0, 1, 2, \dots,$ it is possible to define a new inner product on the Hilbert space in question for certain real values of s in a "critical interval" $0 < s < c_p$ (for $p = 0$, $c_p = \frac{3}{2}$, while for $p = 1, 2, \dots,$ $c_p = \frac{1}{2}$). With respect to this new inner product, $\pi(p, s)$ is unitary. The collection of unitary representations

$$\hat{G}_c = \left\{ \pi(p, s) : 0 < s < 3/2 \text{ if } p = 0, 0 < s < \tfrac{1}{2} \text{ if } p = 1, 2, \dots \right\}$$

obtained in this fashion are all irreducible and pairwise inequivalent. They constitute the representations in the complementary series of G. The results of [**8**, p. 391] show that $\pi(p, s) \simeq \nu_{p,\sigma}$ for $\sigma = \frac{1}{4} - s^2$.

Representations of the form $\nu_{0,\sigma}$ are called class one representations since they contain the trivial representation when restricted to K.

(B). *The representations* $\pi^{+}_{p,q}$, $p = \frac{1}{2}, 1, \dots ; q = p, p - 1, \dots, 1$ or $\frac{1}{2}$. These are the representations arising from Figure 2.

(C). *The representations* $\pi^{-}_{p,q}$, $p = \frac{1}{2}, 1, \dots ; q = p, p - 1, \dots, 1$ or $\frac{1}{2}$. These are the representations arising from Figure 3.

The collection $\hat{G}_d = \{\pi^{\pm}_{p,q} : q \neq \frac{1}{2}\}$ is called the discrete series of G. The representations in $\hat{G}_d$ are the only irreducible unitary representations of G which are square integrable (in fact they are integrable for $q \geqslant \frac{5}{2}$). Global realizations of these representations are given in [**8**].

(D). *The representations* $\pi_{p,0}$, $p = 1, 2, \dots$. These are the representations arising from Figure 4. We denote this collection by $\hat{G}_e$.

The above representations are single- or double-valued representations of G' according to whether $p = 0, 1, 2, \dots$ or not.

Thus we may write $\hat{G} = \hat{G}_i \cup \hat{G}_r \cup \hat{G}_c \cup \hat{G}_d \cup \hat{G}_e \cup \{I\}$ where I is the trivial representation of G. Plancherel measure μ_G on $\hat{G}$ is concentrated in $\hat{G}_i \cup \hat{G}_r \cup \hat{G}_d$ (see [**10**, Vol. II]). On $\hat{G}_i \cup \hat{G}_r$ it is a continuous measure (see [**4**]) while $\mu_G(\pi) > 0$ for each $\pi \in \hat{G}_d$.

3. Restrictions to MA. In this section we determine the restriction of each $\pi \in \hat{G}_p \cup \hat{G}_d$ to the subgroup $C = MA$. For $\pi \in \hat{G}_p$, $(\pi)_C$ was determined in [**7**] by using Mackey's subgroup theorem. We summarize these results in Theorem 1. For $\pi \in \hat{G}_d$, we find $(\pi)_C$ by working on the Lie algebra $\mathfrak{g}$ of G and using Dixmier's realizations of these representations. Throughout this paper we denote by ${}^{A}R$ the regular representation of A acting on $L^2(A)$.

THEOREM 1. *Let* $\rho = \pi(n, iy) \in \hat{G}_p$. *Then*

$$(\rho)_C \simeq \begin{cases} \{\sigma^0 \oplus 3\sigma^1 \oplus \cdots \oplus (2n+1)(\sigma^n \oplus \sigma^{n+1} \oplus \ldots)\} \times {}^A R & \text{if } n = 0, 1, \ldots, \\ \{2\sigma^{1/2} \oplus 4\sigma^{3/2} \oplus \cdots \oplus (2n+1)(\sigma^n \oplus \sigma^{n+1} \oplus \ldots)\} \times {}^A R & \text{if } n = \frac{1}{2}, \frac{3}{2}, \ldots. \end{cases}$$

PROOF. One easily computes that the actions of M and A on V are given by: $m_u v_x m_u^{-1} = v_{ux\bar{u}}$ and $a_t v_x a_t^{-1} = v_{e^t x}$, i.e., MA acts on V by rotations and dilations. Thus there are just two orbits in V under this action–the zero orbit and everything else. Taking v_x with $x = i$ as a representative of the nonzero orbit, one easily computes that the stability group corresponding to this orbit is

$$M_0 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = t_\theta : \theta \in \mathbf{R} \right\}.$$

Writing $\lambda_n(t_\theta) = (e^{i\theta})^{2n}$, we have that $\hat{M}_0 = \{\lambda_n : n = 0, \pm\frac{1}{2}, \pm 1, \ldots\}$. One sees easily, using the isomorphism $M \approx SU(2)$ and the realization of $\sigma^k \in \hat{M}$ given on p. 110 of **[9]**, that

$$(\sigma^k)_{M_0} \simeq \lambda_{-k} \oplus \lambda_{-k+1} \oplus \cdots \oplus \lambda_k \quad \text{for } k = 0, \pm\tfrac{1}{2}, \pm 1, \ldots.$$

According to the results of **[7]**, $S = \{v_0, v_i\}$ is, up to a set of measure zero, a cross-section for V/MA as well as for $P \setminus G/MA$ and $(\rho)_C \simeq \mathrm{Ind}_{M_0}^C(\sigma^n)_{M_0} \simeq \mathrm{Ind}_{M_0}^M(\sigma^n)_{M_0} \times {}^A R$. Theorem 1 now follows by using compact reciprocity and the fact that $(\sigma^n)_{M_0} \simeq \lambda_{-n} \oplus \cdots \oplus \lambda_n$.

We now turn our attention to $\hat{G}_d$ (we will also include the representations in $\hat{G}_r$ $(q = \frac{1}{2})$ since our arguments hold for these representations as well). We first note that for $\rho \in \hat{G}_d$, the results of **[7]**, combined with our knowledge of Plancherel measure on $\hat{C} = \hat{M} \times \hat{A}$, imply that

$$(\rho)_C \simeq \bigoplus_{\hat{M}} n(\sigma, \rho)\sigma \times \int_{\hat{A}}^{\oplus} \tau \, d\mu_A(\tau) \simeq \bigoplus_{\hat{M}} n(\sigma, \rho)\sigma \times {}^A R$$

and so the fact that ${}^A R$ always occurs in $(\rho)_C$ for $\rho \in \hat{G}_d$ will come as no surprise in our next theorem. The main task will be to determine the multiplicities $n(\sigma, \rho) \in \{0, 1, \ldots, \infty\}$.

THEOREM 2. *For* $n = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ *and* $q = n, n-1, \ldots, 1$ *or* $\frac{1}{2}$,

$$\left(\pi_{n,q}^{\pm}\right)_C \simeq \{\sigma^q \oplus 2\sigma^{q+1} \oplus \cdots \oplus (n-q+1)(\sigma^n \oplus \sigma^{n+1} \oplus \ldots)\} \times {}^A R.$$

PROOF. We will consider the case of $\pi_{n,q}^-$. The case $\pi_{n,q}^+$ is similar. If we let $\rho = \pi_{n,q}^-$, then $H_\rho = \bigoplus_\Gamma V^{k,k'}$ (the Hilbert space direct sum) where Γ has the form indicated in Figure 3, §2. We will use the realization of ρ as a representation of $\mathfrak{g}$ given in **[2]** acting on $H = \Sigma_\Gamma V^{k,k'}$ (the algebraic sum). We begin by using our knowledge of $(\sigma^{k,k'})_M$ to decompose each $V^{k,k'}$ into M-invariant subspaces. Since $(\sigma^{k,k'})_M \simeq \sigma^k \otimes \sigma^{k'} \simeq \bigoplus_{|k-k'|}^{k+k'} \sigma^j$ we know that, under the change of basis (using the Clebsch-Gordan coefficients) described in **[9**, pp. 174–194**]**, $V^{k,k'} = V^{|k-k'|} \oplus \ldots \oplus V^{k+k'}$ where $(\rho)_M$ acts on each V^j as σ^j. Fortunately we do not need to describe this change of basis (via the Clebsch-Gordan coefficients on each $V^{k,k'}$) explicitly. We only need to know that such a change exists and that these new basis elements can be chosen so as to transform under $(\rho)_m$ according to the formulas on p. 13 of [**2**]–in particular, each new basis element in V^j is an eigenvector of $\rho(X)$ for $X = \frac{1}{2}(L - L')$, the eigenvalues running from $i(-j), i(-j+1), \ldots, i(j)$.

Next we rearrange the spaces $V^{k,k'}$ appearing in Dixmier's description of $(\rho)_K$ into $n - q + 1$ columns $C_q, C_{q+1}, \ldots, C_n$ where $C_j = \{V^{k,k'}: k - k' = j\}$ for $j = q, q+1, \ldots, n$. So in our new basis, we may view H as being the algebraic sum of the following spaces (a box with a j in it denotes a copy of V^j):

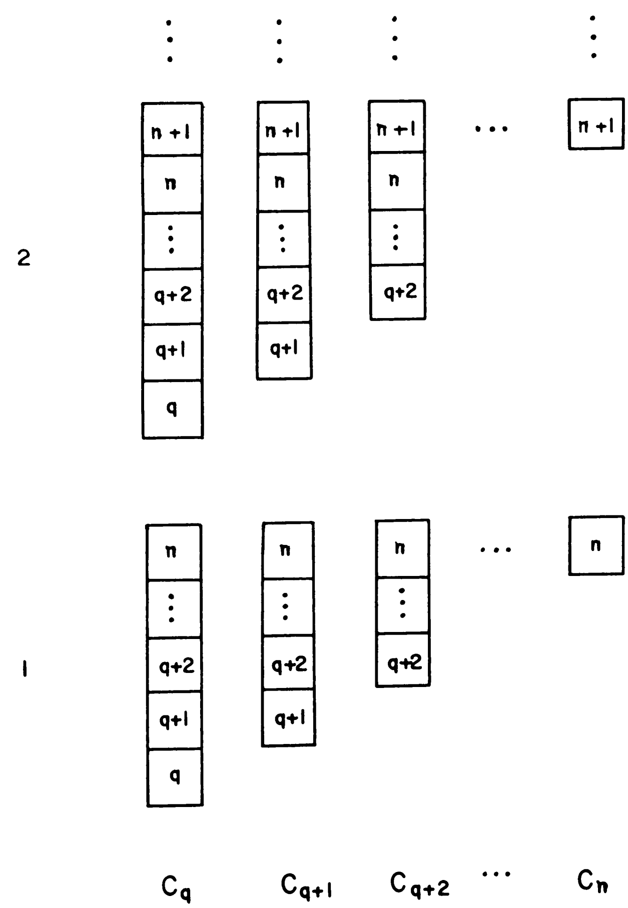

FIGURE 5

We note that under this arrangement, the "closest neighbors" of $V^{k,k'}$ are the four (or less in certain cases) horizontal and vertical neighbors of $V^{k,k'}$ with $V^{k+1/2,k'+1/2}$, $V^{k-1/2,k'-1/2}$ lying above and below $V^{k,k'}$, respectively, and $V^{k+1/2,k'-1/2}$, $V^{k-1/2,k'+1/2}$ lying to the right and left of $V^{k,k'}$, respectively. We will denote by $[p]^{j,k}$ the copy of V^p in column j at level k. We note that $[p]$'s occur in the first $p - q + 1$ columns for $q \leq p \leq n$, in each of the $n - q + 1$ columns for

$p \geqslant n$, and that there are no $[p]$'s for $p < q$ or $p - q \equiv \frac{1}{2}$ (mod 1). This is essentially where the multiplicities alluded to in our theorem come from.

We now investigate the action of $H = \frac{1}{2}(X_\gamma + X_{-\gamma})$, as given on p. 15 of **[2]**, on these basis vectors. In light of the KAK decomposition, the fact that ρ is an irreducible representation of G (recall that $A = \{\exp(tH): t \in \mathbf{R}\}$), and the fact that K leaves each connected set of squares invariant, we expect that the H-action will be somewhat complicated. We do note the following:

(1) If $f \in H$ is a (finite) linear combination of basis elements of the form $f^{k,k'}_{\nu,\nu'}$ (in the "old" basis) and we select a nonzero component of f of the form $cf^{k,k'}_{\nu,\nu'}$ with $k - k'$ minimal and then $k + k'$ maximal (i.e., as far left and up on Figure 5 as possible), then $\rho(H)(cf^{k,k'}_{\nu,\nu'})$ will always have a nonzero component in $V^{k+1/2,k'+1/2}$ which cannot be cancelled out by any of the other components of $\rho(H)f$. So $\rho(H)f \neq 0$ whenever $0 \neq f \in H$ and hence the operator $\rho(H)$ will always be 1-1 on H.

(2) If we let $\rho^+(H)f^{k,k'}_{\nu,\nu'}$ denote the components of $\rho(H)f^{k,k'}_{\nu,\nu'}$ which lie in $V^{k+1/2,k'+1/2} \oplus V^{k+1/2,k'-1/2}$ and $\rho^-(H)f^{k,k'}_{\nu,\nu'}$ denote the components of $\rho(H)f^{k,k'}_{\nu,\nu'}$ which lie in $V^{k-1/2,k'+1/2} \oplus V^{k-1/2,k'-1/2}$, then $\rho(H) = \rho^+(H) + \rho^-(H)$. The same argument as in (1) shows that $\rho^+(H)$ is also 1-1 on H. One may have $\rho^-(H)f = 0$ for $0 \neq f \in H$ (as we shall see).

(3) For $X \in \mathfrak{m}$, $[X, H] = 0$ and so $\rho(X)\rho(H) = \rho(H)\rho(X)$ on H. From this one easily sees that $\rho(X)\rho^{\pm}(H)f^{k,k'}_{\nu,\nu'} = \rho^{\pm}(H)\rho(X)f^{k,k'}_{\nu,\nu'}$ for each $f^{k,k'}_{\nu,\nu'} \in H$ and so both $\rho^+(H)$ and $\rho^-(H)$ commute with the action of $(\rho)_{\mathfrak{m}}$ as well.

(4) If $W \subseteq H$ is a direct sum of a finite number of distinct copies of $[p]$ for a fixed $p \in \{q, q+1, \dots\}$ and $\tilde{W} = \rho(H)W = \{\rho(H)w: w \in W\}$, then $\tilde{W}$ is invariant under $(\rho)_{\mathfrak{m}}$ [from (3)] and in fact the action of $(\rho)_{\mathfrak{m}}$ on $\tilde{W}$ is equivalent to that of $(\rho)_{\mathfrak{m}}$ on W [$\rho(H)$ is the bijection from W to $\tilde{W}$ that intertwines them]. Similarly, the action of $(\rho)_{\mathfrak{m}}$ on such a W is equivalent to that of $(\rho)_{\mathfrak{m}}$ on $\rho^+(H)W$. The same is true for such a W when the action of $\rho^-(H)$ is 1-1.

(5) Applying (3) with $X = \frac{1}{2}(L - L') \in \mathfrak{m}$, we see that if $v \in H$ is an eigenvector of $\rho(X)$ with eigenvalue c, then $\rho(H)v$, $\rho^+(H)v$, and $\rho^-(H)v$ (whenever $\rho^-(H)v \neq 0$) will also be eigenvectors of $\rho(X)$ with eigenvalue c.

(6) For $p \in \{q, q+1, \dots\}$ we know that $\rho(H)[p]^{j,k} \subseteq V^{j,k+1} \oplus V^{j+1,k} \oplus V^{j-1,k} \oplus V^{j,k-1}$. From (4) we know that $(\rho)_{\mathfrak{m}}$ acts on $\rho(H)[p]^{j,k}$ as σ^p, and since $(\rho)_{\mathfrak{m}}$ also leaves each of the spaces $V^{j,k+1}$, $V^{j+1,k}$, $V^{j-1,k}$, $V^{j,k-1}$ invariant, we must have

$$\rho(H)[p]^{j,k} \subseteq [p]^{j,k+1} \oplus [p]^{j+1,k} \oplus [p]^{j-1,k} \oplus [p]^{j,k-1},$$

i.e., for each $f \in [p]^{j,k}$, $\rho(H)f$ is a linear combination of vectors in the closest neighboring $[p]$'s of $[p]^{j,k}$. From (5) we also know that if $f \in [p]^{j,k}$ is an eigenvector of $\frac{1}{2}(L - L')$ with eigenvalue c, then $\rho(H)f$ will be a linear combination of eigenvectors in each of the closest neighboring $[p]$'s of $[p]^{j,k}$ with the same eigenvalue c. Similarly, $\rho^+(H)[p]^{j,k}$ is contained in $[p]^{j,k+1} \oplus [p]^{j+1,k}$, and if $f \in [p]^{j,k}$ is an eigenvector of $\frac{1}{2}(L - L')$ with eigenvalue c, each of the components of $\rho^+(H)f$ in $[p]^{j,k+1}$, $[p]^{j+1,k}$ will also be an eigenvector of $\frac{1}{2}(L - L')$ with eigenvalue c.

(7) Since $i\rho(H)$ is hermitian, we have that $\langle \rho(H)f, g\rangle = -\langle f, \rho(H)g\rangle$ for $f, g \in H$. A routine calculation (first on basis vectors of the form $f_{\nu,\nu'}^{k,k'}$ and then by linearity to all of H) shows that $\langle \rho^+(H)f, g\rangle = -\langle f, \rho^-(H)g\rangle$ for $f, g \in H$.

(8) If for a fixed $f \in H$ we let $e_n = \rho^n(H)f$ for $n = 0, 1, \ldots$, $(e_n) = \mathbf{C}e_n$, $n = 0, 1, \ldots$, and $H_f = \sum_{n=0}^{\infty}(e_n)$, then H_f is invariant under the action of $\mathfrak{a}$ and $\rho(H)e_n = e_{n+1}$. We claim that the action of $(\rho)_{\mathfrak{a}}$ on H_f is equivalent to the regular representation of $\mathfrak{a}$. To see this we first note that, according to the results of [7, p. 202], the restriction to A of the discrete series representation D_1 of SL(2, $\mathbf{R}$) is unitarily equivalent to ${}^A R$. The action of D_1 as a representation of the Lie algebra of SL(2, $\mathbf{R}$) is described in [5, p. 119] ($D_1 \simeq d\pi_1$). D_1 acts on $H' = \sum_{n\geq 2, n\equiv 0 \,(\mathrm{mod}\, 2)}(\phi_n)$, $H = \frac{1}{2}(E^+ + E^-)$, $\mathfrak{a} = \{\exp(tH)\colon\ t \in \mathbf{R}\}$, and $D_1(H)\phi_n = \frac{1}{2}\{-(n-2)\phi_{n-2} + (n+2)\phi_{n+2}\}$. Let $\tilde{e}_n = D_1^n(H)\phi_2$ for $n = 0, 1, \ldots$. Then $H' = \sum_{n=0}^{\infty}(\tilde{e}_n)$ and $D_1(H)\tilde{e}_n = \tilde{e}_{n+1}$. Certainly the action of $(D_1)_{\mathfrak{a}}$ on H' (which is equivalent to the regular representation of $\mathfrak{a}$) is equivalent to the action of $(\rho)_{\mathfrak{a}}$ on H_f and so our claim follows.

We now fix $p \in \{q, q+1, \ldots\}$. Then $[p]$ will appear for the first time in column q at level

$$s = \begin{cases} 1 & \text{for } q \leq p \leq n, \\ p - n + 1 & \text{for } p > n. \end{cases}$$

We set $D_1 = [p]^{q,s}$. We let D_2 denote the direct sum of all $[p]$'s which appear in the main diagonal above D_1, D_3 denote the direct sum of all $[p]$'s which appear in the main diagonal above D_2, etc. So for $p \geq n$,

$$\begin{aligned}
D_2 &= [p]^{q,p-n+2} \oplus [p]^{q+1,p-n+1}, \\
D_3 &= [p]^{q,p-n+3} \oplus [p]^{q+1,p-n+2} \oplus [p]^{q+2,p-n+1}, \qquad \text{for } k \leq n-q+1, \\
D_k &= [p]^{q,p-n+k} \oplus [p]^{q+1,p-n+k-1} \oplus \cdots \oplus [p]^{q+k-1,p-n+1}, \\
D_k &= [p]^{q,p-n+k} \oplus [p]^{q+1,p-n+k-1} \oplus \cdots \oplus [p]^{n,p-n+k-(n-q)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } k \geq n-q+1.
\end{aligned}$$

For $q \leq p \leq n$, $[p]$'s appear only in the first $p - q + 1$ columns and so for $k \leq p - q + 1$,

$$D_k = [p]^{q,k} \oplus [p]^{q+1,k-1} \oplus \cdots \oplus [p]^{q+k-1,1},$$

while for $k \geq p - q + 1$,

$$D_k = [p]^{q,k} \oplus \cdots \oplus [p]^{p,k-(p-q)}.$$

We note that for $q \leq p \leq n$, D_1 has one copy of $[p]$, D_2 has (possibly) two copies of $[p]$, and so forth until we get to D_{p-q+1} which has $p - q + 1$ copies of $[p]$; for $k \geq p - q + 1$, D_k will always have $p - q + 1$ ($\leq n - q + 1$) copies of $[p]$. For $p \geq n$, D_1 starts with one copy of $[p]$, D_2 has two copies, and so forth until we get to D_{n-q+1} which has $n - q + 1$ copies of $[p]$; for $k \geq n - q + 1$, D_k will always have $n - q + 1$ copies of $[p]$. So for $q \leqslant p \leqslant n$, we have

$$\dim(D_{k+1}) = \dim(D_k) + (2p + 1) \quad \text{for } k < p - q + 1 \text{ and}$$
$$\dim(D_{k+1}) = \dim(D_k) \quad \text{for } k \geqslant p - q + 1,$$

while for $p \geqslant n$, we have

$$\dim(D_{k+1}) = \dim(D_k) + (2p + 1) \quad \text{for } k < n - q + 1 \text{ and}$$
$$\dim(D_{k+1}) = \dim(D_k) \quad \text{for } k \geqslant n - q + 1.$$

We also note that for $k \geqslant 2$, $\rho(H)D_k \subseteq D_{k-1} \oplus D_k$ with $\rho^-(H)D_k \subseteq D_{k-1}$ and $\rho^+(H)D_k \subseteq D_{k+1}$ while $\rho(H)D_1 = \rho^+(H)D_1 \subseteq D_2$.

We now establish the inductive step of the procedure we shall use for obtaining a copy of $m_p\sigma^p \times {}^AR$ in $(\rho)_C$ where

$$m_p = \begin{cases} p - q + 1 & \text{for} \quad q \leqslant p \leqslant n, \\ n - q + 1 & \text{for} \quad p \geqslant n. \end{cases}$$

Claim 1. If $D_k = W_k^1 \oplus \cdots \oplus W_k^j$ where each W_k^i, $i = 1, \ldots, j$, is an invariant subspace of D_k on which $(\rho)_{\mathfrak{m}}$ acts as σ^p and we set $W_{k+1}^i = \rho^+(H)W_k^i$ for $i = 1, \ldots, j$, then:

(i) If $\dim(D_{k+1}) = \dim(D_k)$, we have that $D_{k+1} = W_{k+1}^1 \oplus \cdots \oplus W_{k+1}^j$ where $(\rho)_{\mathfrak{m}}$ acts on each W_{k+1}^i as σ^p.

(ii) If $\dim(D_{k+1}) = \dim(D_k) + (2p + 1)$, we have that $D_{k+1} = W_{k+1}^1 \oplus \cdots \oplus W_{k+1}^j \oplus W_{k+1}^{j+1}$ where W_{k+1}^{j+1} is the orthogonal complement in D_{k+1} of $\rho^+(H)D_k$ and $(\rho)_{\mathfrak{m}}$ acts on each W_{k+1}^i as σ^p for $i = 1, \ldots, j + 1$.

Proof of Claim 1. (i) We know from note (2) that $\rho^+(H)D_k$ will be a subspace of D_{k+1} with the same dimension as D_k and so, in this case, $\rho^+(H)D_k = D_{k+1}$. From note (3), we have that $\rho^+(H)$ commutes with the action of $\mathfrak{m}$ and so each W_{k+1}^i will be invariant under the action of $(\rho)_{\mathfrak{m}}$ and this action will be equivalent to that of σ^p.

(ii) As in (i), we have $\rho^+(H)D_k = W_{k+1}^1 \oplus \cdots \oplus W_{k+1}^j$, but in this case, $\rho^+(H)D_k$ is properly contained in D_{k+1}. Certainly $D_{k+1} = \rho^+(H)D_k \oplus W_{k+1}^{j+1}$ and the action of $(\rho)_{\mathfrak{m}}$ on W_{k+1}^{j+1} is also equivalent to σ^p.

Claim 2. (i) If $\dim(D_{k+1}) = \dim(D_k)$ and $D_k = W_k^1 \oplus \cdots \oplus W_k^j$, $D_{k+1} = W_{k+1}^1 \oplus \cdots \oplus W_k^j$ are as in Claim 1, then for each $i = 1, \ldots, j$, $\rho^-(H)W_{k+1}^i = W_k^i$.

(ii) If $\dim(D_{k+1}) = \dim(D_k) + (2p + 1)$ and $D_k = W_k^1 \oplus \cdots \oplus W_k^j$, $D_{k+1} = W_{k+1}^1 \oplus \cdots \oplus W_{k+1}^j \oplus W_{k+1}^{j+1}$ are as in Claim 1, then for $i = 1, \ldots, j$, $\rho^-(H)W_{k+1}^i = W_k^i$ while $\rho^-(H)W_{k+1}^{j+1} = 0$.

Proof of Claim 2. (i) Let $g \in W_{k+1}^i$ with $g = \rho^+(H)f$ for $f \in W_k^i$. Then by note (7), $\langle g, g\rangle = \langle \rho^+(H)f, g\rangle = -\langle f, \rho^-(H)g\rangle$ and so $\rho^-(H)g \neq 0$ whenever $0 \neq g \in W_{k+1}^i$ and $\rho^-(H)W_{k+1}^i = W_k^i$.

(ii) As in (i), $\rho^-(H)W_{k+1}^i = W_k^i$ for $i = 1, \ldots, j$. Now let $g \in W_{k+1}^{j+1}$ and $f \in D_k$. From note (7), we have $\langle f, \rho^-(H)g\rangle = -\langle \rho^+(H)f, g\rangle = 0$ by the definition of W_{k+1}^{j+1}.

Now for $p = q, q + 1, \ldots$ we begin with $D_1 = W_1^1$ and use induction to decompose each D_k into subspaces having the properties stated in our claims. So for $q \leqslant p \leqslant n$,

$$D_k = W_k^1 \oplus \cdots \oplus W_k^k \quad \text{for } k < p - q + 1,$$

$$D_k = W_k^1 \oplus \cdots \oplus W_k^{p-q+1} \quad \text{for } k \geqslant p - q + 1.$$

For $p \geqslant n$,

$$D_k = W_k^1 \oplus \cdots \oplus W_k^k \quad \text{for } k < n - q + 1$$

$$D_k = W_k^1 \oplus \cdots \oplus W_k^{n-q+1} \quad \text{for } k \geqslant n - q + 1.$$

For $q \leqslant p \leqslant n$, we let

$$H_p^1 = \sum_{k=1}^{\infty} W_k^1, \quad H_p^2 = \sum_{k=2}^{\infty} W_k^2, \ldots, \quad H_p^{p-q+1} = \sum_{k=p-q+1}^{\infty} W_k^{p-q+1}.$$

Each of these $p - q + 1$ subspaces of H is invariant under the action of $(\rho)_{\mathfrak{m}\oplus\mathfrak{a}}$, and by (8), $(\rho)_{\mathfrak{m}\oplus\mathfrak{a}}$ acting on each of these spaces is equivalent to $\sigma^p \times {}^AR$. So for $q \leqslant p \leqslant n$, we see that $\sigma^p \times {}^AR$ will occur in $(\rho)_{\mathfrak{m}\oplus\mathfrak{a}}$ with multiplicity $p - q + 1$. For $p \geqslant n$, we let

$$H_p^1 = \sum_{k=1}^{\infty} W_k^1, \ldots, H_p^{n-q+1} = \sum_{k=n-q+1}^{\infty} W_k^{n-q+1}.$$

Each of these $n - q + 1$ subspaces of H is invariant under the action of $(\rho)_{\mathfrak{m}\oplus\mathfrak{a}}$ and equivalent to $\sigma^p \times {}^AR$. So for $p \geqslant n$, we see that $\sigma^p \times {}^AR$ will occur in $(\rho)_{\mathfrak{m}\oplus\mathfrak{a}}$ with multiplicity $n - q + 1$ and Theorem 2 has been proven.

4. Tensor products of principal series. In this section we combine the results of the previous section with those of [7] to determine the decomposition of the tensor product of two principal series representations of G. If $\sigma^m, \sigma^n \in \hat{M}$ with $m \geqslant n$, $\lambda^{is}, \lambda^{iy} \in \hat{A}$, then (as described in [7]) we have, after a routine application of Mackey's tensor product theorem, that $\pi(m, is) \otimes \pi(n, iy) \simeq \operatorname{Ind}_C^G(\sigma^m \otimes \sigma^n) \times \lambda^{i(s+y)}$. Thus the problem of decomposing the tensor product of two principal series representations reduces to knowing the decomposition $\sigma^m \otimes \sigma^n \simeq \bigoplus_{m-n}^{m+n} \sigma^k$ and then the decomposition of $\operatorname{Ind}_C^G L$ for all $L \in \hat{C}$. Since inducing from C is independent of the character on A (see [7, p. 188]), the latter decomposition is known once one knows how to decompose $\operatorname{Ind}_C^G L$ for μ_C-almost all $L \in \hat{C}$. By the Mackey-Anh reciprocity theorem, the problem of finding $\operatorname{Ind}_C^G L$ for μ_C-almost all $L \in \hat{C}$ is equivalent to finding $(\pi)_C$ for μ_G-almost all $\pi \in \hat{G}$, i.e., for almost every principal series representation and every discrete series representation of G.

THEOREM 3. *Let* $\sigma^n \in \hat{M}$ *and* $\tau \in \hat{A}$. *Then* $\operatorname{Ind}_C^G \sigma^n \times \tau \simeq T_c \oplus T_d$ *where* T_c *is a continuous direct integral with respect to Plancherel measure on* $\hat{G}$ *of representations* $\pi(k, s)$ *from the principal series of* G *with* $k + n \equiv 0 \pmod 1$ *and* T_d *is a discrete direct sum of discrete series representations* $\pi_{k,q}^{\pm}$ *with* $k + n \equiv 0 \pmod 1$ *and* 1 *or* $\frac{3}{2} \leqslant q \leqslant \min\{k, n\}$ (*and so* $T_d = 0$ *for* $n = 0, \frac{1}{2}$). *For* 0 *or* $\frac{1}{2} \leqslant k \leqslant n$, *the representation* $\pi(k, s)$ ($s \geqslant 0$ *for* $k = 0, 1, \ldots$ *and* $s > 0$ *for* $k = \frac{1}{2}, \frac{3}{2}, \ldots$) *occurs in* T_c *with multiplicity* $2k + 1$ *while the representation* $\pi_{k,q}^+ \oplus \pi_{k,q}^-$, $3/2$ *or* $1 \leqslant q \leqslant k$, *occurs in* T_d *with multiplicity* $k - q + 1$. *For* $k \geqslant n$, *the representation* $\pi(k, s)$ *occurs in* T_c *with multiplicity* $2n + 1$ *while the representation* $\pi_{k,q}^+ \oplus \pi_{k,q}^-$, 1 *or* $3/2 \leqslant q \leqslant n$, *occurs in* T_d *with multiplicity* $n - q + 1$. *For* $k \geqslant n$ *and* $n < q \leqslant k$, *the multiplicity of* $\pi_{k,q}^+ \oplus \pi_{k,q}^-$ *in* T_d *is zero.*

PROOF. This is immediate from the Mackey-Anh reciprocity theorem in conjunction with Theorems 1 and 2.

REMARKS. (1) We have found tables of the following form helpful in generating the multiplicities which appear in Theorem 3. In these tables, the index k appears between the double lines while the index q appears in the left-hand column below these double lines. The numbers above the double lines indicate multiplicities of principal series representations occurring in T_c while those below the double lines indicate multiplicities of the sum $\pi^+_{k,q} \oplus \pi^-_{k,q}$ in T_d. So, for example, when $n = 0, 1, \ldots$ we would write

		1	3	5	7	...	2n-1	2n+1	2n+1	...
q	k	0	1	2	3	...	n-1	n	n+1	...
1			1	2	3	...	n-1	n	n	...
2				1	2	...	n-2	n-1	n-1	...
3					1	...	n-3	n-2	n-2	...
.							.	.	.	
.							.	.	.	
.							.	.	.	
n-1							1	2	2	...
n								1	1	...

(2) Since $\pi(n, s) \otimes \pi(0, t) \simeq \operatorname{Ind}_C^G \sigma^n \times \lambda^{i(s+t)}$, Theorem 3 already provides a decomposition for this tensor product. We note that in contrast to the SL(2, **R**) case (see [7, p. 204]), the tensor product of two principal series representations need not contain discrete series representations in its decomposition (i.e., $T_d = 0$)–as in the case for $n = 0$ and $n = \frac{1}{2}$. For $n = 0$, each principal series representation $\pi(k, s)$ for $k = 0, 1, \ldots$ and $s \geqslant 0$ will occur with multiplicity 1 while for $n = \frac{1}{2}$, each $\pi(k, s)$ for $k = \frac{1}{2}, \frac{3}{2}, \ldots$ and $s > 0$ will occur with multiplicity 2.

THEOREM 4. *Let $\pi(m, \cdot)$ and $\pi(n, \cdot)$ be two principal series representations of $G = \mathrm{Spin}(4, 1)$ with $m \geqslant n$. Then $\pi(m, \cdot) \otimes \pi(n, \cdot) \simeq T_c \oplus T_d$ where T_c is a continuous direct integral with respect to Plancherel measure on $\hat{G}$ of representations $\pi(k, s)$ from the principal series of G with $k + m + n \equiv 0$ (mod 1) and T_d is a discrete direct sum of discrete series representations $\pi^{\pm}_{k,q}$ with $k + m + n \equiv 0$ (mod 1) and 1 or $3/2 \leqslant q \leqslant \min\{k, m + n\}$ (and so $T_d = 0$ for $m = n = 0$ or $m = \frac{1}{2}$, $n = 0$). The representation $\pi(k, s)$ occurs in T_c with multiplicity*

$$
\begin{aligned}
&(2k + 1)(2n + 1) && \text{for } 0 \text{ or } \tfrac{1}{2} \leqslant k \leqslant m - n, \\
&(2k + 1)(2n + 1) - \sum_{i=1}^{h} 2i && \text{for } k = m - n + h, h = 1, 2, \ldots, 2n, \\
&(2m + 1)(2n + 1) && \text{for } k \geqslant m + n.
\end{aligned}
$$

The representation $\pi^+_{k,q} \oplus \pi^-_{k,q}$ *occurs in* T_d *with multiplicity*

$$(k-q+1)(2n+1) \qquad \textit{for } 0 \textit{ or } \tfrac{1}{2} \leqslant k \leqslant m-n,\ 1 \textit{ or } 3/2 \leqslant q \leqslant k,$$

$$(k-q+1)(2n+1) - \sum_{i=1}^{h} i \qquad \textit{for } k = m-n+h,\ h = 1, 2, \dots, 2n,\ 1 \textit{ or } 3/2 \leqslant q \leqslant m-n,$$

$$(k-q+1)(2n+1-j) - \sum_{i=1}^{h-j} i \qquad \textit{for } k = m-n+h,\ h = 1, 2, \dots, 2n,\ q = m-n+j,\ j = 1, \dots, h,$$

$$(m-q+1)(2n+1) \qquad \textit{for } k \geqslant m+n,\ 1 \textit{ or } 3/2 \leqslant q \leqslant m-n,$$

$$(2n-j+1)^2 - \sum_{i=1}^{2n-j} i \qquad \textit{for } k \geqslant m+n,\ q = m-n+j,\ j = 1, \dots, 2n,$$

$$0 \qquad \textit{for } k > m+n,\ m+n < q \leqslant k.$$

PROOF. We have that $\pi(m, \cdot) \otimes \pi(n, \cdot) \simeq \bigoplus_{m-n}^{m+n} \operatorname{Ind}_C^G \sigma^p \times (\cdot)$ where each of the $2n+1$ representations in the latter sum may be decomposed by using Theorem 3. Note that multiplicities of principal and discrete series representations appearing in a given $\operatorname{Ind}_C^G \sigma^p \times (\cdot)$ increase (by 2 for principal series representations and 1 for discrete series representations) until $k = p$ and that for $k \geqslant p$ they are "at rest". So, for 0 or $\frac{1}{2} \leqslant k \leqslant m-n$, $\pi(k, s)$ will occur in the decomposition of $\pi(m, \cdot) \otimes \pi(n, \cdot)$ with multiplicity $(2n+1)(2k+1)$. For $m-n+1 \leqslant k \leqslant m+n$, the multiplicities of $\pi(k, s)$ in $k-(m-n)$ of the representations $\operatorname{Ind}_C^G \sigma^p \times (\cdot), p = m-n, \dots, m+n$, are at rest and so the actual multiplicity of $\pi(k, s)$ in T_c will be $(2n+1)(2k+1) - \sum_{i=1}^{k-(m-n)} 2i$. For $k = m+n$, we will be subtracting $\sum_{i=1}^{2n} 2i = 2n(2n+1)$ and so the actual multiplicity will be $(2(m+n)+1)(2n+1) - 2n(2n+1) = (2m+1)(2n+1)$. The multiplicity of $\pi(k, s)$ in our decomposition will then be $(2m+1)(2n+1)$ for all $k \geqslant m+n$. Similar reasoning applies for the multiplicities of discrete series representations appearing in T_d except we must also account for the fact that when $k = m-n+h$, $h = 1, \dots, 2n$, and $q > m-n$ with $q = m-n+j$, $j = 1, \dots, h$, there are only $2n+1-j$ contributions with $h-j$ of them "at rest".

REMARKS. (1) One can use tables similar to those described in Remark 1 of Theorem 3 to generate the above multiplicities. For example, in the case of $m+n \equiv 0 \pmod 1$ we would write the integers $0, 1, \dots, m+n$ in between a pair of lines to denote the k-index of a representation appearing in our decomposition. Next we would write the integers $1, \dots, m+n$ in a column to the left and below the above row to denote the q-index of discrete series representations appearing in this decomposition. The multiplicities for the $\pi(k, s)$'s for $0 \leqslant k \leqslant m+n$ are then placed above their appropriate k-indices. They are easy to generate from the theorem. Now if we denote the multiplicity of $\pi^+_{k,q} \oplus \pi^-_{k,q}$ in T_d by $m_{k,q}$, then $m_{q,q} = 2n+1$ for $1 \leqslant q \leqslant m-n$, while $m_{q,q} = 2n+1-j$ for $q = m-n+j$, $j = 1, \dots, 2n$. These numbers are now placed along the main diagonal in our table. The remaining $m_{k,q}$'s for $k > q$ may now be filled in from our knowledge of the $m_{q,q}$'s by noting (after checking each of the cases in Theorem 4) that $m_{k+1,q} - m_{k,q} = m_{k+1,k+1}$ for each k, and so $m_{q+j,q} = m_{q,q} + m_{q+1,q+1} + \cdots + m_{q+j,q+j}$ for $1 \leq j \leq m + n - q$. The multiplicities $m_{k,q}$ may now be filled in one row at a time.

For example, $\pi(5, \cdot) \otimes \pi(2, \cdot)$ might be exhibited as

		5	15	25	35	43	49	53	55	...
q	k	0	1	2	3	4	5	6	7	...
1			5	10	15	19	22	24	25	...
2				5	10	14	17	19	20	...
3					5	9	12	14	15	...
4						4	7	9	10	...
5							3	5	6	...
6								2	3	...
7									1	...

(2) The above decomposition for $m = n = 0$ has also been obtained in [3, p. 202].

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DEPARTMENT OF MATHEMATICS, MIDDLEBURY COLLEGE, MIDDLEBURY, VERMONT 05753